

# THE DIRICHLET PROBLEM FOR A CLASS OF HESSIAN TYPE EQUATIONS

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**ABSTRACT.** We are concerned with the Dirichlet problem for a class of Hessian type equations. Applying some new methods we are able to establish the  $C^2$  estimates for an approximating problem under essentially optimal structure conditions. Based on these estimates, the existence of classical solutions is proved.

**Keywords:** Hessian equations, interior second order estimates, classical solutions.

## 1. INTRODUCTION

Let  $\Omega$  be a bounded domain in  $\mathbb{R}^n$  ( $n \geq 2$ ) with smooth boundary  $\partial\Omega$ . In this paper, we are concerned with the regularity for solutions of the Dirichlet problem

$$(1.1) \quad \begin{cases} f(\lambda[D^2u + \gamma\Delta uI]) = \psi & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{cases}$$

where  $\gamma \geq 0$  is a constant,  $I$  is the unit matrix and  $\lambda[D^2u + \gamma\Delta uI] = (\lambda_1, \dots, \lambda_n)$  denote the eigenvalues of the matrix  $\{D^2u + \gamma\Delta uI\}$ .

Following [1],  $f \in C^2(\Gamma) \cap C(\bar{\Gamma})$  is assumed to be defined in an open convex symmetric cone  $\Gamma$ , with vertex at the origin and

$$\Gamma \supseteq \Gamma_n \equiv \{\lambda \in \mathbb{R}^n : \text{each component } \lambda_i > 0\},$$

and to satisfy the following structure conditions:

$$(1.2) \quad f_i \equiv \frac{\partial f}{\partial \lambda_i} > 0 \text{ in } \Gamma, 1 \leq i \leq n,$$

$$(1.3) \quad f \text{ is concave in } \Gamma,$$

and

$$(1.4) \quad f > 0 \text{ in } \Gamma, f = 0 \text{ on } \partial\Gamma.$$

A function  $u \in C^2(\Omega)$  is called *admissible* if  $\lambda[D^2u + \gamma\Delta uI] \in \bar{\Gamma}$ . According to [1], condition (1.2) ensures that equation (1.1) is degenerate elliptic for admissible solutions. While (1.3) implies that the function  $F$  defined by  $F[A] = f(\lambda[A])$  to be concave for  $A \in \mathcal{S}^{n \times n}$  with  $\lambda[A] \in \Gamma$ , where  $\mathcal{S}^{n \times n}$  is the set of  $n$  by  $n$  symmetric matrices.

We assume that  $\psi \geq 0$  in  $\Omega$ , so the equation (1.1) is degenerate when  $\gamma = 0$ . In this paper, there are no geometric restrictions to  $\partial\Omega$  being made. Instead, we

assume that there exists a subsolution  $\underline{u} \in C^2(\bar{\Omega})$  satisfying  $\lambda(D^2\underline{u} + \gamma\Delta\underline{u}I) \in \Gamma$  on  $\bar{\Omega}$  and

$$(1.5) \quad \begin{cases} f(\lambda(D^2\underline{u} + \gamma\Delta\underline{u}I)) \geq \psi & \text{in } \Omega, \\ \underline{u} = \varphi & \text{on } \partial\Omega. \end{cases}$$

**Theorem 1.1.** *Let  $\gamma > 0$ ,  $\psi \in C^\infty(\bar{\Omega})$  and  $\varphi \in C^\infty(\partial\Omega)$ . Under (1.2)-(1.5), there exists a unique admissible solution  $u \in C^\infty(\bar{\Omega})$  of (1.1).*

We first introduce our procedure to prove Theorem 1.1. By (1.4), there exists a positive constant  $\varepsilon_0$  such that

$$(1.6) \quad f(\lambda(D^2\underline{u} + \gamma\Delta\underline{u}I)) \geq \varepsilon_0 \text{ on } \bar{\Omega}$$

since  $\lambda(D^2\underline{u} + \gamma\Delta\underline{u}I) \in \Gamma$ . We shall establish the *a priori*  $C^2$  estimates independent of  $\varepsilon$  for admissible solutions of the approximating problem

$$(1.7) \quad \begin{cases} f(\lambda(D^2u_\varepsilon + \gamma\Delta u_\varepsilon I)) = \psi + \varepsilon\eta(\psi) & \text{in } \Omega, \\ u_\varepsilon = \varphi & \text{on } \partial\Omega, \end{cases}$$

where  $\eta \in C^\infty[0, \infty)$  satisfies

$$\eta(t) = \begin{cases} 1 & \text{if } t \in [0, \frac{\varepsilon_0}{4}], \\ 0 & \text{if } t \in [\frac{\varepsilon_0}{2}, \infty), \end{cases}$$

$0 \leq \eta \leq 1$ ,  $|\eta'| \leq C\varepsilon_0^{-1}$  and  $|\eta''| \leq C\varepsilon_0^{-2}$ . It follows that, by (1.6),

$$f(\lambda(D^2\underline{u} + \gamma\Delta\underline{u}I)) \geq \psi + \varepsilon\eta(\psi)$$

provided  $\varepsilon \leq \frac{\varepsilon_0}{2}$  and obviously,  $\psi + \varepsilon\eta(\psi) \geq \min\{\varepsilon, \varepsilon_0/4\} > 0$ .

We shall use the techniques of Guan [7] (see [8] and [9] also) to establish such estimates. As usual, the main difficulty is from the boundary estimates of pure normal second order derivative for which we use the strategy of Ivochkina, Trudinger and Wang [11] whose idea is originally from Krylov [13, 14, 15, 16] where the Bellman equations are studied. A key step is the construction of barrier functions in which the existence of  $\underline{u}$  plays an important role (see Theorem 5.1).

The presence of  $\gamma > 0$  is crucial to the interior estimates for second derivatives. An interesting question is to establish the weak interior estimates (see [11]) when  $\gamma = 0$ .

For the case that  $\psi \geq \psi_0 > 0$ , the existence of smooth solutions to the Dirichlet problem (1.1) with  $\gamma = 0$  was established by Caffarelli, Nirenberg and Spruck [1] under additional assumptions on  $f$  in a domain  $\Omega$  satisfying that there exists a sufficiently large number  $R > 0$  such that, at every point  $x \in \partial\Omega$ ,

$$(1.8) \quad (\kappa_1, \dots, \kappa_{n-1}, R) \in \Gamma,$$

where  $\kappa_1, \dots, \kappa_{n-1}$  are the principal curvatures of  $\partial\Omega$  with respect to the interior normal. Their work was further developed and simplified by Trudinger [17].

Guan considered the Hessian equations of the form

$$(1.9) \quad f(\lambda[\nabla^2 u + \gamma\Delta u g + sdu \otimes du - \frac{t}{2}|\nabla u|^2 g + A]) = \psi(x, u, \nabla u)$$

on a Riemannian manifold with metric  $g$  with  $\psi > 0$ , which is arising from conformal geometry (see [4] and [5]). In these papers Guan also assumed that  $f$  is homogenous of degree one which implies that the equation (1.9) is strictly elliptic. It would be

interesting to prove Theorem 1.1 for the general form (1.9) on manifolds when  $\psi \geq 0$  without any additional conditions on  $f$ . The case that  $\gamma = 0$  seems more complicated. In a recent work [7], Guan proved Theorem 1.1 under (1.2)-(1.5) when  $\gamma = 0$  and  $\psi \geq \psi_0 > 0$ . Another interesting question would be whether we can get a viscosity solution in  $C^{1,1}(\bar{\Omega})$  for  $\gamma = 0$  when  $\psi \geq 0$ .

It was shown in [1] that using (1.8) and the condition that for every  $C > 0$  and every compact set  $K$  in  $\Gamma$  there is a number  $R = R(C, K)$  such that

$$(1.10) \quad f(R\lambda) \geq C \text{ for all } \lambda \in K$$

one can construct admissible strict subsolutions of equation (1.1) with  $\gamma = 0$ . Obviously  $\Gamma \subset \{\lambda \in \mathbb{R}^n : \sum \lambda_i > 0\}$  and we have  $\Delta u \geq 0$  for any admissible function  $u$ . So we can construct an admissible strict subsolution of (1.1) when  $\gamma \geq 0$  satisfying (1.5) under (1.8) and (1.10) by the same way.

Typical examples are given by  $f = \sigma_k^{1/k}$  and  $f = (\sigma_k/\sigma_l)^{1/(k-l)}$ ,  $1 \leq l < k \leq n$ , defined in the Gårding cone

$$\Gamma_k = \{\lambda \in \mathbb{R}^n : \sigma_j(\lambda) > 0, j = 1, \dots, k\},$$

where  $\sigma_k$  are the elementary symmetric functions

$$\sigma_k(\lambda) = \sum_{i_1 < \dots < i_k} \lambda_{i_1} \dots \lambda_{i_k}, \quad k = 1, \dots, n.$$

The case when  $f = \sigma_n^{1/n}$  (the Monge-Ampère equation) and  $\gamma = 0$  was studied by Guan, Trudinger and Wang [10] and they obtained the  $C^{1,1}$  regularity as  $\psi^{1/(n-1)} \in C^{1,1}(\bar{\Omega})$ . It would be an interesting problem to show whether the result can be improved for the  $f = \sigma_k^{1/k}$  (see [11]).

The rest of this paper is organized as follows. In Section 2, we prove Theorem 1.1 provided the  $C^2$  estimates for (1.7) is established.  $C^1$  estimate is treated in Section 3. The interior second order estimate is proved in Section 4. In section 5, the estimates for second derivatives are established.

## 2. BEGINNING OF PROOF

In this Section we explain how to prove Theorem 1.1 when the second order estimates for (1.7) are established. Let  $u_\varepsilon \in C^4(\bar{\Omega})$  be the admissible solution of (1.7). For simplicity we shall use the notations  $U^\varepsilon = D^2 u_\varepsilon + \gamma \Delta u_\varepsilon I$  and  $\underline{U} = D^2 \underline{u} + \gamma \Delta \underline{u} I$ . Following the literature, unless otherwise noted, we denote throughout this paper

$$F^{ij}[U^\varepsilon] = \frac{\partial F}{\partial U_{ij}^\varepsilon}[U^\varepsilon], \quad F^{ij,kl}[U^\varepsilon] = \frac{\partial^2 F}{\partial U_{ij}^\varepsilon \partial U_{kl}^\varepsilon}[U^\varepsilon].$$

The matrix  $\{F^{ij}\}$  has eigenvalues  $f_1, \dots, f_n$  and is positive definite by assumption (1.2), while (1.3) implies that  $F$  is a concave function of  $U_{ij}^\varepsilon$  (see [1]). Moreover, when  $U^\varepsilon$  is diagonal so is  $\{F^{ij}\}$ , and the following identities hold

$$F^{ij} U_{ij}^\varepsilon = \sum f_i \lambda_i, \quad F^{ij} U_{ik}^\varepsilon U_{kj}^\varepsilon = \sum f_i \lambda_i^2, \quad \lambda[U^\varepsilon] = (\lambda_1, \dots, \lambda_n).$$

Suppose  $\gamma > 0$  and we have proved that there exists a constant independent of  $\varepsilon$  such that

$$(2.1) \quad |u_\varepsilon|_{C^2(\bar{\Omega})} \leq C.$$

Therefore, by the concavity of  $F$ ,

$$F^{ij}[U_\varepsilon](A\delta_{ij} - U_{ij}^\varepsilon) \geq F[AI] - F[U_\varepsilon] \geq c_0 > 0$$

by fixing  $A$  sufficiently large. On the other hand,  $-F^{ij}U_{ij}^\varepsilon \leq C \sum F^{ii}$  by (2.1). Then we get

$$\sum F^{ii} \geq \frac{c_0}{A+C} > 0.$$

Note that

$$\left\{ \frac{\partial F}{\partial u_{ij}^\varepsilon}[U_\varepsilon] \right\} = \{F^{ij}[U_\varepsilon]\} + \gamma \sum F^{ii} I \geq \frac{\gamma c_0}{A+C} I.$$

Thus, there exists uniform constants  $0 < \lambda_0 \leq \Lambda_0 < \infty$  such that

$$\lambda_0 I \leq \left\{ \frac{\partial F}{\partial u_{ij}^\varepsilon}[U_\varepsilon] \right\} \leq \Lambda_0 I.$$

Hence Evans-Krylov theory (see [2] and [12]) assures a bound  $M$  independent of  $\varepsilon$  such that

$$|u_\varepsilon|_{C^{2,\alpha}(\bar{\Omega})} \leq M,$$

for some constant  $\alpha \in (0, 1)$ . The higher regularity can be derived by the Schauder theory (see [3] for example). Using standard method of continuity, we can obtain the existence of smooth solution to (1.7). By sending  $\varepsilon$  to zero (taking a subsequence if necessary), we can prove Theorem 1.1.

In the following sections, we may drop the subscript  $\varepsilon$  when there is no possible confusion.

### 3. THE GRADIENT ESTIMATES

In this section, we consider the gradient estimates for the admissible solution to (1.7). We first observe that  $\lambda[U] \in \Gamma \subset \{\sum \lambda_i > 0\}$  and therefore,

$$(3.1) \quad \text{tr}[U] = (1 + n\gamma)\Delta u > 0.$$

Thus we have by the maximum principle that

$$\underline{u} \leq u \leq h \text{ in } \bar{\Omega}$$

where  $h$  is the harmonic function in  $\Omega$  with  $h = \varphi$  on  $\partial\Omega$ . Then we obtain

$$(3.2) \quad \sup_{\bar{\Omega}} |u| + \sup_{\partial\Omega} |Du| \leq C,$$

for some positive constant  $C$  independent of  $\varepsilon$ .

To establish the global gradient estimates, we assume that  $|Du|e^\phi$  achieves a maximum at an interior point  $x_0 \in \Omega$ , where  $\phi$  is a function to be determined. We may assume  $D^2u$  and  $\{F^{ij}\}$  are diagonal at  $x_0$  by rotating the coordinates if necessary. Then at  $x_0$  where the function  $\log |Du| + \phi$  attains its maximum, we have

$$(3.3) \quad \frac{u_k u_{ki}}{|Du|^2} + \phi_i = 0$$

and

$$(3.4) \quad \frac{u_k u_{kii} + u_{ki} u_{ki}}{|Du|^2} - 2 \frac{(u_k u_{ki})^2}{|Du|^4} + \phi_{ii} \leq 0$$

for each  $i = 1, \dots, n$ . Differentiating the equation (1.7), we get, at  $x_0$ ,

$$(3.5) \quad F^{ii}u_{kii} + \gamma\Delta u_k \sum F^{ii} = \psi_k + \varepsilon\eta'\psi_k.$$

It follows that

$$(3.6) \quad F^{ii}u_k u_{kii} + \gamma u_k \Delta u_k \sum F^{ii} \geq -C|Du|.$$

Note that

$$(3.7) \quad \begin{aligned} U_{ii}^2 &= (u_{ii} + \gamma\Delta u)^2 \leq 2u_{ii}^2 + 2\gamma^2(\Delta u)^2 \leq 2u_{ii}^2 + 2n\gamma^2 \sum_j u_{jj}^2 \\ &\leq 2n \max\{\gamma, 1\} (u_{ii}^2 + \gamma \sum_j u_{jj}^2). \end{aligned}$$

Therefore, by (3.3), (3.4), (3.6) and (3.7), we have

$$(3.8) \quad \begin{aligned} &c_0 F^{ii} U_{ii}^2 + |Du|^2 \left( F^{ii} \phi_{ii} + \gamma \Delta \phi \sum F^{ii} \right) \\ &\leq C|Du| + 2|Du|^2 \left( F^{ii} \phi_i^2 + \gamma |D\phi|^2 \sum F^{ii} \right), \end{aligned}$$

where  $c_0 = (2n \max\{\gamma, 1\})^{-1}$ .

Let  $v = \underline{u} - u + \inf_{\bar{\Omega}}(u - \underline{u}) + 1$  and  $\phi = \frac{\delta v^2}{2}$ , where  $\delta$  is a positive constant to be determined. Choosing  $\delta$  sufficiently small, we can guarantee that

$$\delta - 2\delta^2 v^2 > 0 \text{ on } \bar{\Omega}.$$

Let  $c_1 = \min_{x \in \bar{\Omega}} (\delta - 2\delta^2 v^2(x)) > 0$ . It follows from (3.8) that

$$(3.9) \quad \begin{aligned} &c_0 F^{ii} U_{ii}^2 + |Du|^2 \mathcal{L}(\underline{u} - u) \\ &\leq C|Du| - (\delta - 2\delta^2 v^2) |Du|^2 \left( F^{ii} v_i^2 + \gamma |Dv|^2 \sum F^{ii} \right) \\ &\leq C|Du| - c_1 |Du|^2 \left( F^{ii} v_i^2 + \gamma |Dv|^2 \sum F^{ii} \right). \end{aligned}$$

Write  $\mu(x) = \lambda(D^2 \underline{u}(x) + \gamma \Delta \underline{u}(x)I)$  and note that  $\{\mu(x) : x \in \bar{\Omega}\}$  is a compact subset of  $\Gamma$ . There exists uniform constant  $\beta \in (0, \frac{1}{2\sqrt{n}})$  such that

$$(3.10) \quad \nu_{\mu(x)} - 2\beta \mathbf{1} \in \Gamma_n, \quad \forall x \in \bar{\Omega}$$

where  $\mathbf{1} = (1, \dots, 1) \in \mathbb{R}^n$  and  $\nu_\lambda := Df(\lambda)/|Df(\lambda)|$  is the unit normal vector to the level hypersurface  $\partial\Gamma^{f(\lambda)}$  for  $\lambda \in \Gamma$ . We need the following lemma proved by Guan in [7].

**Lemma 3.1.** *For any fixed  $x \in \bar{M}$ , denote  $\tilde{\mu} = \mu(x)$  and  $\tilde{\lambda} = \lambda(U(x))$ . Suppose that  $|\nu_{\tilde{\mu}} - \nu_{\tilde{\lambda}}| \geq \beta$ . Then there exists a uniform constant  $\theta > 0$  such that*

$$(3.11) \quad \sum f_i(\tilde{\lambda})(\tilde{\mu}_i - \tilde{\lambda}_i) \geq \theta \left( 1 + \sum f_i(\tilde{\lambda}) \right).$$

Now let  $\mu = \lambda(D^2 \underline{u}(x_0) + \gamma \Delta \underline{u}(x_0))$ ,  $\lambda = \lambda(D^2 u(x_0) + \gamma \Delta u(x_0))$  and  $\beta$  as in (3.10). Suppose first that  $|\nu_\mu - \nu_\lambda| \geq \beta$ . Define the linear operator  $\mathcal{L}$  by

$$\mathcal{L}v := F^{ij}v_{ij} + \gamma \Delta v \sum F^{ii}$$

for  $v \in C^2(\Omega)$ . By Lemma 3.1,

$$(3.12) \quad \mathcal{L}(\underline{u} - u) \geq \theta \left( \sum F^{ii} + 1 \right)$$

for some  $\theta > 0$ . Thus, we can obtain a bound  $|Du(x_0)| \leq C/\theta$  from (3.9).

We now consider the case  $|\nu_\mu - \nu_\lambda| < \beta$  which implies  $\nu_\lambda - \beta \mathbf{1} \in \Gamma_n$  and therefore

$$(3.13) \quad F^{ii} \geq \frac{\beta}{\sqrt{n}} \sum F^{kk}, \quad \forall 1 \leq i \leq n.$$

By the concavity of  $F$  we know that

$$(3.14) \quad \mathcal{L}(\underline{u} - u) \geq 0.$$

By the concavity of  $f$  again, when  $|\lambda| \geq R$  for  $R$  sufficiently large, we derive as in [7]

$$(3.15) \quad \begin{aligned} |\lambda| \sum F^{ii} &\geq f(|\lambda| \mathbf{1}) - f(\lambda) + \sum F^{ii} \lambda_i \\ &\geq f(|\lambda| \mathbf{1}) - f(\mu) - |\lambda| \sum F^{ii} \\ &\geq 2b_0 - |\lambda| \sum F^{ii}. \end{aligned}$$

for some uniform positive constant  $b_0$ . Therefore, by (3.13) and (3.15), we find

$$(3.16) \quad \begin{aligned} c_0 F^{ii} U_{ii}^2 + c_1 |Du|^2 F^{ii} v_i^2 &\geq \frac{\beta}{\sqrt{n}} \left( c_0 |\lambda|^2 \sum F^{ii} + \frac{c_1}{2} |Du|^4 \sum F^{ii} \right) \\ &\geq \frac{\sqrt{2c_0 c_1} \beta}{\sqrt{n}} |Du|^2 |\lambda| \sum F^{ii} \\ &\geq c_2 |Du|^2 \end{aligned}$$

provided  $|Du|$  is sufficiently large, where  $c_2 = \frac{\sqrt{2c_0 c_1} \beta b_0}{\sqrt{n}}$ . Thus, from (3.9) and (3.16) we can get a bound  $|Du| \leq C/c_2$ .

Suppose  $|\lambda| \leq R$ . By the concavity of  $F$ , we have (see [9])

$$2R \sum F^{ii} \geq F^{ii} U_{ii} + F(2RI) - F(U) \geq -R \sum F^{ii} + b_1,$$

where  $b_1 = F(2RI) - F(RI) > 0$ . It follows that

$$(3.17) \quad \sum F^{ii} \geq \delta_0 \equiv \frac{b_1}{3R}$$

and

$$c_1 |Du|^2 F^{ii} v_i^2 \geq \frac{c_1 \beta}{2\sqrt{n}} |Du|^4 \sum F^{ii} \geq \frac{c_1 \beta \delta_0}{2\sqrt{n}} |Du|^4$$

provided  $|Du|$  is sufficiently large. We then obtain from (3.9) that  $|Du(x_0)| \leq (2\sqrt{n}C/c_1\beta\delta_0)^{1/3}$ .

Hence we have proved that

$$(3.18) \quad |u|_{C^1(\bar{\Omega})} \leq C$$

for some positive constant  $C$  independent of  $\varepsilon$ .

#### 4. INTERIOR AND GLOBAL ESTIMATES FOR SECOND DERIVATIVES

In this section, we prove the interior second order estimate.

**Theorem 4.1.** *Let  $\gamma > 0$  and  $u \in C^4(\Omega)$  be an admissible solution of (1.7). Then for any  $\Omega' \subset \subset \Omega$ , there exists a constant  $C$  depending on  $\gamma^{-1}$ ,  $d' \equiv \text{dist}(\Omega', \partial\Omega)$ ,  $|u|_{C^1(\bar{\Omega})}$  and other known data such that*

$$(4.1) \quad \sup_{\bar{\Omega}'} |D^2 u| \leq C.$$

*Proof.* Let

$$W = \max_{x \in \bar{\Omega}, |\xi|=1} \zeta(x) e^{\phi(x)} D_{\xi\xi} u(x)$$

where  $\zeta$  and  $\phi$  are functions to be determined with  $\zeta$  satisfying

$$(4.2) \quad 0 \leq \zeta \leq 1, \quad |D\zeta| \leq a_0, \quad |D^2\zeta| \leq a_0 \quad \text{on } \bar{\Omega}.$$

Assume that  $W$  is achieved at  $x_0 \in \Omega$  and  $\xi_0 = e_1 = (1, 0, \dots, 0)$ . We may also assume that  $D^2u$  is diagonal at  $x_0$ . We have, at  $x_0$  where the function  $\log u_{11} + \log \zeta + \phi$  attains its maximum,

$$(4.3) \quad \frac{u_{11i}}{u_{11}} + \frac{\zeta_i}{\zeta} + \phi_i = 0 \quad \text{for each } i = 1, \dots, n,$$

$$(4.4) \quad F^{ii} \left\{ \frac{u_{11ii}}{u_{11}} - \left( \frac{u_{11i}}{u_{11}} \right)^2 - \left( \frac{\zeta_i}{\zeta} \right)^2 + \frac{\zeta_{ii}}{\zeta} + \phi_{ii} \right\} \leq 0.$$

and

$$(4.5) \quad \frac{\Delta u_{11}}{u_{11}} - \sum_i \left( \frac{u_{11i}}{u_{11}} \right)^2 - \sum_i \left( \frac{\zeta_i}{\zeta} \right)^2 + \frac{\Delta \zeta}{\zeta} + \Delta \phi \leq 0.$$

Differentiating equation (1.7) twice, by the concavity of  $F$ , we obtain at  $x_0$ ,

$$(4.6) \quad F^{ii} u_{ii11} + \gamma (\Delta u)_{11} \sum F^{ii} = \psi_{11} + \varepsilon (\eta' \psi_{11} + \eta'' \psi_1^2) \geq -C.$$

Let

$$\phi = \frac{\delta |Du|^2}{2},$$

where  $\delta > 0$  is an undetermined constant. By straightforward calculation, we have

$$\phi_i = \delta u_i u_{ii}$$

and

$$\phi_{ii} = \delta u_{ii}^2 + \delta u_j u_{jii}.$$

Note that

$$(4.7) \quad F^{ii} u_j u_{jii} + \gamma u_j \Delta u_j \sum F^{ii} = u_j (\psi_j + \varepsilon \eta' \psi_j) \geq -C$$

and

$$(4.8) \quad \phi_i^2 \leq C \delta^2 u_{ii}^2.$$

We have

$$(4.9) \quad \mathcal{L}\phi \geq \delta F^{ii} u_{ii}^2 + \gamma \delta \sum u_{jj}^2 \sum F^{ii} - C\delta.$$

Combining (4.3)-(4.9), we get

$$(4.10) \quad \begin{aligned} 0 \geq & -\frac{C}{u_{11}} - C\delta + (\delta - C\delta^2) F^{ii} u_{ii}^2 \\ & + \gamma (\delta - C\delta^2) \sum u_{jj}^2 \sum F^{ii} - \frac{C}{\zeta^2} \sum F^{ii}. \end{aligned}$$

Choose  $\delta$  sufficiently small such that  $\delta - C\delta^2 > 0$ . Let  $\lambda = \lambda(D^2u(x_0) + \gamma \Delta u(x_0))$ . By (3.7), we find that

$$|\lambda|^2 = \sum U_{ii}^2 \leq 2n^2 \left( \max\{\gamma, 1\} \right)^2 \sum u_{jj}^2.$$

Thus, it follows from (4.10) that

$$(4.11) \quad 0 \geq -\frac{C}{u_{11}} - C\delta + 2c_3|\lambda|^2 \sum F^{ii} - \frac{C}{\zeta^2} \sum F^{ii},$$

where

$$c_3 = \frac{1}{4}(\delta - C\delta^2)\gamma n^{-2} \left( \max\{\gamma, 1\} \right)^{-2} > 0.$$

By (3.15) and (4.11), we have

$$(4.12) \quad 0 \geq \left( b_0 c_3 |\lambda| - \frac{C}{u_{11}} - C\delta \right) + \left( c_3 |\lambda|^2 - Cb^2 - \frac{C}{\zeta^2} \right) \sum F^{ii}$$

provided  $|\lambda|$  is sufficiently large. It follows that  $|\lambda|\zeta(x_0) \leq C$ .

The function  $\zeta$  may now be chosen as a cutoff function satisfying  $\zeta \equiv 1$  on  $\Omega' \subset\subset \Omega$  and  $|D\zeta| \leq C/d'$ ,  $|D^2\zeta| \leq C/d'^2$ . Then

$$|D^2u|\zeta \leq C \text{ on } \bar{\Omega}$$

and (4.1) holds.  $\square$

*Remark 4.2.* We remark that in the proof of Theorem 4.1 we do not need the existence of  $\underline{u}$ .

In the proof of Theorem 4.1, setting  $\zeta \equiv 1$ , we can prove the following maximal principle.

**Theorem 4.3.** *Let  $u \in C^4(\bar{\Omega})$  be an admissible solution of (1.7). Then*

$$(4.13) \quad \sup_{\bar{\Omega}} |D^2u| \leq C(1 + \sup_{\partial\Omega} |D^2u|),$$

where  $C$  depends on  $\gamma^{-1}$ ,  $|u|_{C^1(\bar{\Omega})}$  and other known data.

We are interested in the case that  $\gamma = 0$ . Now we prove (4.13) under the existence of  $\underline{u}$  satisfying (1.5) and we will see that the constant  $C$  would not depend on  $\gamma^{-1}$  when  $\gamma$  is small.

**Theorem 4.4.** *Suppose (1.2)-(1.5) hold. Let  $u \in C^4(\bar{\Omega})$  be an admissible solution of (1.7). Then we have*

$$(4.14) \quad \sup_{\bar{\Omega}} |D^2u| \leq C \max\{\gamma, 1\} (1 + \sup_{\partial\Omega} |D^2u|),$$

for some constant  $C$  depending on  $|u|_{C^1(\bar{\Omega})}$ ,  $|\underline{u}|_{C^2(\bar{\Omega})}$  and other known data. In particular, if  $0 \leq \gamma \leq 1$ , (4.13) holds for the constant  $C$  depending on  $|u|_{C^1(\bar{\Omega})}$ ,  $|\underline{u}|_{C^2(\bar{\Omega})}$  and other known data.

*Proof.* In the proof of Theorem 4.1, let  $\zeta \equiv 1$  and  $\phi = \frac{\delta}{2}|Du|^2 + b(\underline{u} - u)$ , where  $\delta$  and  $b$  are positive constants to be chosen. Note that

$$\phi_i = \delta u_i u_{ii} + b(\underline{u} - u)_i$$

and

$$\phi_{ii} = \delta u_{ii}^2 + \delta u_j u_{jii} + b(\underline{u} - u)_{ii}.$$

We have

$$(4.15) \quad \phi_i^2 \leq C\delta^2 u_{ii}^2 + Cb^2.$$

Therefore, by (4.7),

$$(4.16) \quad \mathcal{L}\phi \geq \delta F^{ii} u_{ii}^2 + \gamma \delta \sum u_{jj}^2 \sum F^{ii} - C\delta + b\mathcal{L}(\underline{u} - u).$$



We can derive from (4.3)-(4.6) and (4.15) that

$$(4.17) \quad \mathcal{L}\phi \leq \frac{C}{u_{11}} + C\delta^2(F^{ii}u_{ii}^2 + \gamma \sum u_{jj}^2 \sum F^{ii}) + Cb^2 \sum F^{ii}.$$

Combining (4.16) and (4.17), we obtain

$$(4.18) \quad (\delta - C\delta^2)(F^{ii}u_{ii}^2 + \gamma \sum u_{jj}^2 \sum F^{ii}) + b\mathcal{L}(\underline{u} - u) \leq C\delta + \frac{C}{u_{11}} + Cb^2 \sum F^{ii}.$$

We may assume that  $\delta$  is sufficiently small such that  $(\delta - C\delta^2) > \delta/2$ . let  $\mu = \lambda(D^2\underline{u}(x_0) + \gamma\underline{u}(x_0))$ ,  $\lambda = \lambda(D^2u(x_0) + \gamma\underline{u}(x_0))$ . As in the gradient estimates, we consider two cases: (i)  $|\nu_\mu - \nu_\lambda| < \beta$  and (ii)  $|\nu_\mu - \nu_\lambda| \geq \beta$ , where  $\beta$  is as in (3.10).

In case (i), we see that (3.13) holds. Thus, by (3.7), (4.18), (3.13) and (3.14), we have

$$(4.19) \quad (2n \max\{\gamma, 1\})^{-1} \frac{\delta\beta}{2\sqrt{n}} |\lambda|^2 \sum F^{ii} \leq C\delta + \frac{C}{u_{11}} + Cb^2 \sum F^{ii}.$$

We may assume  $|\lambda| \geq R$  for  $R$  sufficiently large such that (3.15) holds. Therefore, by (3.15) and (4.19), we have

$$(4.20) \quad (2n \max\{\gamma, 1\})^{-1} \left( \frac{\delta\beta}{4\sqrt{n}} |\lambda|^2 \sum F^{ii} + \frac{\delta\beta b_0}{4\sqrt{n}} |\lambda| \right) \leq C\delta + \frac{C}{u_{11}} + Cb^2 \sum F^{ii}.$$

It follows that

$$|\lambda| \leq \frac{8n\sqrt{n} \max\{\gamma, 1\}}{\delta\beta} \max \left\{ Cb, \frac{C\delta}{b_0} + \frac{C}{Rb_0} \right\}.$$

Note that we do not determine  $\delta$  and  $b$  right now.

In case (ii), we can choose  $b$  sufficiently small such that  $\theta b > Cb^2$ , where  $\theta$  is as in (3.11). We can choose such  $b$  and a smaller  $\delta$  such that  $\theta b > C\delta$ . Applying Lemma 3.1, we can derive from (4.18) that

$$u_{11} \leq \frac{C}{\theta b - C\delta}$$

and (4.14) holds.  $\square$

## 5. BOUNDARY ESTIMATES FOR SECOND DERIVATIVES

In this section we consider the estimates for the second order derivatives on the boundary  $\partial\Omega$ . As usual, the construction of barrier functions plays a key role. For any fixed  $x_0 \in \Omega$ , we may assume that  $x_0$  is the origin of  $\mathbb{R}^n$  with the positive  $x_n$  axis in the interior normal direction to  $\partial\Omega$  at the origin. Let  $d(x)$  be the distances from  $x \in \bar{\Omega}$  to  $\partial\Omega$ , and set

$$\Omega_\delta = \{x \in \Omega : |x| < \delta\}.$$

Suppose near the origin, the boundary  $\partial\Omega$  is represented by

$$(5.1) \quad x_n = \rho(x') = \frac{1}{2} \sum_{\alpha, \beta < n} B_{\alpha\beta} x_\alpha x_\beta + O(|x'|^3)$$

for some  $C^\infty$  smooth function  $\rho$ , where  $x' = (x_1, \dots, x_{n-1})$ . For  $x \in \partial\Omega$  near the origin, let

$$T_\alpha = T_\alpha(x) = \partial_\alpha + \sum_{\beta < n} B_{\alpha\beta}(x_\beta \partial_n - x_n \partial_\beta), \quad \text{for } \alpha < n$$

and  $T_n = \partial_n$ . We have (see [1])

$$\mathcal{L}T_\alpha u = T_\alpha(\psi + \varepsilon\eta(\psi)).$$

It follows that

$$(5.2) \quad |\mathcal{L}T_\alpha(u - \varphi)| \leq C \left(1 + \sum F^{ii}\right)$$

and

$$(5.3) \quad |T_\alpha(u - \varphi)| \leq C|x|^2 \text{ on } \partial\Omega \cap \bar{\Omega}_\delta \text{ for } \alpha < n$$

when  $\delta$  is sufficiently small since  $u = \varphi$  on  $\partial\Omega$ .

To proceed we choose smooth unit orthonormal vector fields  $e_1, \dots, e_n$  in  $\Omega_\delta$  such that when restricted to  $\partial\Omega$ ,  $e_1, \dots, e_{n-1}$  are tangential and  $e_n$  is normal to  $\partial\Omega$ . Let  $e_i(x) = (\xi_1^i(x), \dots, \xi_n^i(x))$ ,  $\nabla_i u = \xi_k^i D_k u$ ,  $\nabla_{ij} u = \xi_l^i \xi_k^j D_{kl} u$  and  $\nabla^2 u = \{\nabla_{ij} u\}$  in  $\Omega_\delta$ . We may assume  $\xi_j^i(0) = \delta_{ij}$ . In particular,  $\lambda(D^2 u) = \lambda(\nabla^2 u)$  and

$$f(\lambda(\nabla^2 u + \gamma \Delta u I)) = f(\lambda(D^2 u + \gamma \Delta u I)).$$

By straightforward calculations, we have

$$(5.4) \quad |\mathcal{L}\nabla_k(u - \varphi)| \leq C \left(1 + \sum F^{ii} + \sum f_i |\hat{\lambda}_i| + \gamma |\hat{\lambda}| \sum F^{ii}\right),$$

where  $\hat{\lambda} = \lambda(D^2 u) = \lambda(\nabla^2 u)$ . Let  $\hat{F}^{ij} = \xi_l^i \xi_k^j F^{kl}$ . We see that  $\{\hat{F}^{ij}\}$  is positive definite with eigenvalues  $f_1, \dots, f_n$  and when  $\nabla^2 u$  is diagonal so is  $\{\hat{F}^{ij}\}$ .

We shall use the following barrier function

$$(5.5) \quad \Psi = \frac{1}{\delta^2} \left( A_1(u - \underline{u}) + td - \frac{N}{2} d^2 + A_3 |x|^2 \right) - A_2(u - \underline{u}) - A_4 \sum_{l < n} |\nabla_l(u - \varphi)|^2,$$

where  $A_1, A_2, A_3, A_4, t$  and  $N$  are positive constants satisfying  $A_1 > 2A_2$ .

**Theorem 5.1.** *Suppose that (1.2)-(1.5) hold. Let  $h \in C(\bar{\Omega}_\delta)$  satisfy  $h \leq \bar{C}|x|^2$  on  $\bar{\Omega}_\delta \cap \partial\Omega$  and  $h \leq \bar{C}$  on  $\bar{\Omega}_\delta$ . Then for any positive constant  $K$  there exist uniform positive constants  $t, \delta$  sufficiently small, and  $A_1, A_2, A_3, N$  sufficiently large such that  $\Psi \geq h$  on  $\partial\Omega_\delta$  and*

$$(5.6) \quad \mathcal{L}\Psi \leq -K \left(1 + \sum F^{ii}\right) \text{ in } \Omega_\delta.$$

*Proof.* Let  $v = td - \frac{Nd^2}{2}$ . Firstly, we note that

$$(5.7) \quad \begin{aligned} \mathcal{L}v &= (t - Nd)F^{ij}(d_{ij} + \gamma \Delta d \delta_{ij}) - NF^{ij}(d_i d_j + \gamma |Dd|^2 \delta_{ij}) \\ &\leq C_0(t + Nd) \sum F^{ii} - NF^{ij} d_i d_j - \gamma N \sum F^{ii} \end{aligned}$$

since  $|Dd| \equiv 1$ . Similar to Proposition 2.19 in [6], we have

$$(5.8) \quad \hat{F}^{ij} \nabla_{il} u \nabla_{jl} u \geq \frac{1}{2} \sum_{i \neq r} f_i \hat{\lambda}_i^2$$

and

$$(5.9) \quad \sum_{l < n} \sum_{k=1}^n (\nabla_{lk} u)^2 \geq \frac{1}{2} \sum_{i \neq r} \widehat{\lambda}_i^2$$

for some index  $r$ . It follows from (5.4), (5.8) and (5.9) that

$$(5.10) \quad \begin{aligned} \sum_{l < n} \mathcal{L} |\nabla_l(u - \varphi)|^2 &= 2 \sum_{l < n} F^{ij} \left( (\nabla_l(u - \varphi))_i (\nabla_l(u - \varphi))_j \right. \\ &\quad \left. + \sum_{l < n} \gamma |D \nabla_l(u - \varphi)|^2 \delta_{ij} \right) + 2 \nabla_l(u - \varphi) \mathcal{L} \nabla_l(u - \varphi) \\ &\geq -C \left( 1 + \sum F^{ii} + \sum f_i |\widehat{\lambda}_i| + \gamma |\widehat{\lambda}| \sum F^{ii} \right) \\ &\quad + \sum_{l < n} \widehat{F}^{ij} \nabla_{il} u \nabla_{jl} u + \frac{\gamma}{2} \sum_{l < n} \sum_{k=1}^n (\nabla_{lk} u)^2 \sum F^{ii} \\ &\geq \frac{1}{2} \sum_{i \neq r} f_i \widehat{\lambda}_i^2 + \frac{\gamma}{4} \sum_{i \neq r} \widehat{\lambda}_i^2 \sum f_i \\ &\quad - C \left( 1 + \sum F^{ii} + \sum f_i |\widehat{\lambda}_i| + \gamma |\widehat{\lambda}| \sum F^{ii} \right). \end{aligned}$$

For any  $x \in \Omega_\delta$ , let  $\mu = \lambda(D^2 \underline{u}(x) + \gamma \Delta \underline{u}(x) I)$ ,  $\lambda = \lambda(D^2 u(x) + \gamma \Delta u(x) I)$  and  $\beta$  be as in (3.10). We consider two cases: (i)  $|\nu_\mu - \nu_\lambda| < \beta$  and (ii)  $|\nu_\mu - \nu_\lambda| \geq \beta$ .

In case (i), we see that (3.13) holds. It follows that

$$(5.11) \quad \sum_{i \neq r} f_i \widehat{\lambda}_i^2 \geq \frac{\beta}{\sqrt{n}} \sum_{i \neq r} \widehat{\lambda}_i^2 \sum f_i$$

and

$$(5.12) \quad \mathcal{L} v \leq -\frac{N\beta}{2\sqrt{n}} \sum f_i$$

provided  $t$  and  $\delta$  are sufficiently small since  $Dd \equiv 1$ .

We first assume  $|\lambda| \geq R$  for  $R$  sufficiently large. If  $\widehat{\lambda}_r \leq 0$ , we have  $\sum_{i \neq r} \widehat{\lambda}_i > -\widehat{\lambda}_r$  since  $\Delta u > 0$ . It follows that

$$\sum_{i \neq r} \widehat{\lambda}_i^2 \geq c_0 \widehat{\lambda}_r^2$$

for some uniform constant  $c_0 > 0$ . Therefore, by (3.7), there exists uniform positive constants  $c_1$  and  $c_2$  such that

$$(5.13) \quad \sum_{i \neq r} \widehat{\lambda}_i^2 \geq c_1 \sum \widehat{\lambda}_i^2 \geq c_2 |\lambda|^2.$$

Combining (3.13), (3.14), (3.15), (5.10), (5.12) and (5.13),

$$\begin{aligned}
(5.14) \quad \mathcal{L}\Psi &\leq -\frac{N\beta}{2\sqrt{n}\delta^2} \sum f_i + \frac{CA_3}{\delta^2} \sum f_i - \frac{\beta c_2}{2\sqrt{n}} A_4 |\lambda|^2 \sum f_i \\
&\quad + CA_4 \left(1 + \sum f_i + |\hat{\lambda}| \sum f_i\right) \\
&\leq -\frac{N\beta}{2\sqrt{n}\delta^2} \sum f_i - \frac{\beta c_2 b_0}{4\sqrt{n}} A_4 |\lambda| + \frac{CA_3}{\delta^2} \sum f_i + CA_4 \left(1 + \sum f_i\right) \\
&\leq \left(-\frac{N\beta}{2\sqrt{n}\delta^2} + \frac{CA_3}{\delta^2} + CA_4\right) \sum f_i - A_4
\end{aligned}$$

provided  $|\lambda| > R$  and  $R$  is sufficiently large.

If  $\hat{\lambda}_r > 0$ , we have

$$\begin{aligned}
(5.15) \quad \mathcal{L}u &= F^{ij} u_{ij} + \gamma \triangle u \sum F^{ii} = \sum f_i \hat{\lambda}_i + \gamma \sum \hat{\lambda}_i \sum F^{ii} \\
&\geq -\sum_{i \neq r} f_i |\hat{\lambda}_i| - \gamma \sum_{i \neq r} |\hat{\lambda}_i| \sum F^{ii} + f_r \hat{\lambda}_r + \gamma \hat{\lambda}_r \sum F^{ii}.
\end{aligned}$$

Note that for each  $\sigma > 0$  and each  $1 \leq i \leq n$ ,

$$(5.16) \quad \hat{\lambda}_i^2 \geq 2\sigma |\hat{\lambda}_i| - \sigma^2.$$

It follows that

$$(5.17) \quad \frac{A_4}{2} \sum_{i \neq r} f_i \hat{\lambda}_i^2 \geq 2A_2 \sum_{i \neq r} f_i |\hat{\lambda}_i| - \frac{2A_2^2}{A_4} \sum_{i \neq r} f_i$$

by letting  $\sigma = 2A_2/A_4$  and that

$$(5.18) \quad \frac{A_4}{4} \sum_{i \neq r} \hat{\lambda}_i^2 \geq 2A_2 \sum_{i \neq r} |\hat{\lambda}_i| - \frac{4A_2^2}{A_4}$$

by letting  $\sigma = 4A_2/A_4$ . Therefore, by (5.10), (5.15), (5.17) and (5.18), we find

$$\begin{aligned}
(5.19) \quad &\mathcal{L}\left(A_2(u - \underline{u}) + A_4 \sum_{l < n} |\nabla_l(u - \varphi)|^2\right) \\
&\geq A_4 \left(\frac{1}{2} \sum_{i \neq r} f_i \hat{\lambda}_i^2 + \frac{\gamma}{4} \sum_{i \neq r} \hat{\lambda}_i^2 \sum f_i\right) - CA_2 \sum f_i \\
&\quad + A_2(f_r \hat{\lambda}_r + \gamma \hat{\lambda}_r \sum f_i) - A_2 \left(\sum_{i \neq r} f_i |\hat{\lambda}_i| + \gamma \sum_{i \neq r} |\hat{\lambda}_i| \sum f_i\right) \\
&\quad - CA_4 \left(1 + \sum F^{ii} + \sum f_i |\hat{\lambda}_i| + \gamma |\hat{\lambda}| \sum f_i\right) \\
&\geq (A_2 - CA_4) \left(\sum_{i \neq r} f_i |\hat{\lambda}_i| + \gamma |\hat{\lambda}| \sum f_i\right) - CA_4 \left(1 + \sum f_i\right) \\
&\quad - \left(CA_2 + \frac{2A_2^2}{A_4} + \frac{4A_2^2}{A_4} \gamma\right) \sum f_i.
\end{aligned}$$

Therefore, by (3.13) and (3.15), we have

$$\begin{aligned}
\mathcal{L}\Psi &\leq -\frac{N\beta}{2\sqrt{n}\delta^2} \sum f_i - (A_2 - CA_4) \left( \sum f_i |\widehat{\lambda}_i| + \gamma |\widehat{\lambda}| \sum f_i \right) \\
&\quad + C \left( A_2 + \frac{A_2^2}{A_4} + \frac{A_3}{\delta^2} \right) \sum f_i + CA_4 \left( 1 + \sum f_i \right) \\
&\leq -\frac{N\beta}{2\sqrt{n}\delta^2} \sum f_i - \frac{\beta(A_2 - CA_4)}{\sqrt{n}} |\lambda| \sum f_i \\
&\quad + C \left( A_2 + \frac{A_2^2}{A_4} + \frac{A_3}{\delta^2} \right) \sum f_i + CA_4 \left( 1 + \sum f_i \right) \\
&\leq -\frac{N\beta}{2\sqrt{n}\delta^2} \sum f_i - \frac{\beta(A_2 - CA_4)b_0}{\sqrt{n}} \\
&\quad + C \left( A_2 + \frac{A_2^2}{A_4} + \frac{A_3}{\delta^2} \right) \sum f_i + CA_4 \left( 1 + \sum f_i \right).
\end{aligned} \tag{5.20}$$

Now we assume  $|\lambda| \leq R$ . We see that (3.17) holds. Thus, by (5.12) and (5.10), we have

$$\begin{aligned}
\mathcal{L}\Psi &\leq -\frac{N\beta}{2\sqrt{n}\delta^2} \sum f_i + C \left( A_2 + \frac{A_3}{\delta^2} + A_4 \right) \left( 1 + \sum f_i \right) \\
&\leq -\frac{N\beta}{4\sqrt{n}\delta^2} \sum f_i - \frac{N\beta\delta_0}{4\sqrt{n}\delta^2} + C \left( A_2 + \frac{A_3}{\delta^2} + A_4 \right) \left( 1 + \sum f_i \right).
\end{aligned} \tag{5.21}$$

Now we fix  $A_3 > A_4 > K$  such that

$$A_3 > A_4 \sup_{\Omega} \sum_{l < n} |\nabla_l(u - \varphi)|^2 + \overline{C}. \tag{5.22}$$

In case (ii), we see from Lemma 3.1 that (3.12) holds. We deal with two cases as before:  $\widehat{\lambda}_r > 0$  and  $\widehat{\lambda}_r \leq 0$ .

If  $\widehat{\lambda}_r > 0$ , similar to (5.20), we obtain

$$\begin{aligned}
\mathcal{L}\Psi &\leq -\theta \frac{A_1}{\delta^2} \left( 1 + \sum f_i \right) + C_0(t + Nd) \sum f_i \\
&\quad + \frac{CA_3}{\delta^2} \sum f_i + CA_4 \left( 1 + \sum f_i \right) + C \left( A_2 + \frac{A_2^2}{A_4} \right) \sum f_i \\
&\quad - (A_2 - CA_4) \left( \sum f_i |\widehat{\lambda}_i| + \gamma |\widehat{\lambda}| \sum f_i \right).
\end{aligned} \tag{5.23}$$

If  $\widehat{\lambda}_r \leq 0$ , similar to (5.15),

$$-\mathcal{L}u \geq -\sum_{i \neq r} f_i |\widehat{\lambda}_i| - \gamma \sum_{i \neq r} |\widehat{\lambda}_i| \sum F^{ii} - f_r \widehat{\lambda}_r - \gamma \widehat{\lambda}_r \sum F^{ii}. \tag{5.24}$$

Similar to (5.19), we have, for any  $B > 0$ ,

$$\begin{aligned}
&\mathcal{L} \left( -B(u - \underline{u}) + A_4 \sum_{l < n} |\nabla_l(u - \varphi)|^2 \right) \\
&\geq (B - CA_4) \left( \sum f_i |\widehat{\lambda}_i| + \gamma |\widehat{\lambda}| \sum f_i \right) - CA_4 \left( 1 + \sum f_i \right) \\
&\quad - \left( CB + \frac{2B^2}{A_4} + \frac{4B^2}{A_4} \gamma \right) \sum f_i.
\end{aligned} \tag{5.25}$$

Thus, choosing  $B = A_2$ , we can see from (3.12), (5.7) and (5.25) that

$$(5.26) \quad \begin{aligned} \mathcal{L}\Psi &\leq -\theta\left(\frac{A_1}{\delta^2} - 2A_2\right)\left(1 + \sum f_i\right) + C_0(t + Nd) \sum f_i \\ &\quad + \frac{CA_3}{\delta^2} \sum f_i + CA_4\left(1 + \sum f_i\right) + C\left(A_2 + \frac{A_2^2}{A_4}\right) \sum f_i \\ &\quad - (A_2 - CA_4)\left(\sum f_i |\widehat{\lambda}_i| + \gamma |\widehat{\lambda}| \sum f_i\right). \end{aligned}$$

Now we choose  $A_2 \gg A_4$  such that

$$(5.27) \quad \frac{\beta(A_2 - CA_4)b_0}{\sqrt{n}} - CA_4 > K$$

in (5.20) and  $A_2 - CA_4 > 0$  in (5.23) and (5.26). Next, we fix  $N$  sufficiently large such that

$$(5.28) \quad \frac{N\beta}{2\sqrt{n}\delta^2} - \frac{CA_3}{\delta^2} - CA_4 > K$$

in (5.14),

$$(5.29) \quad \frac{N\beta}{2\sqrt{n}\delta^2} - C\left(A_2 + \frac{A_2^2}{A_4} + \frac{A_3}{\delta^2}\right) - CA_4 > K$$

in (5.20) and

$$(5.30) \quad \frac{N\beta}{4\sqrt{n}\delta^2} \min\{1, \delta_0\} - C\left(A_2 + \frac{A_3}{\delta^2} + A_4\right) > K$$

in (5.21).

We may assume that  $t$  and  $\delta$  is sufficiently small such that  $\theta A_1/\delta^2 > 2C_0(t + Nd)$ . Therefore, in case (ii), by (5.23) and (5.26), we have

$$(5.31) \quad \mathcal{L}\Psi \leq \left(-\frac{A_1\theta}{2\delta^2} + \frac{CA_3}{\delta^2} + CA_2 + C\frac{A_2^2}{A_4} + CA_4\right)\left(1 + \sum f_i\right).$$

Finally, we may choose  $A_1$  large enough to obtain (5.6). Furthermore,  $v \geq 0$  in  $\Omega_\delta$  when  $\delta \leq 2t/N$ . Therefore, we can ensure  $\Psi \geq h$  on  $\partial\Omega_\delta$ .  $\square$

Now we are ready to establish the boundary estimates for second order derivatives. Firstly, it is easy to obtain a bound independent of  $\varepsilon$  for pure tangential second order derivatives on the boundary

$$(5.32) \quad |u_{\xi\eta}|_{C^0(\partial\Omega)} \leq C$$

from the boundary condition in (1.1), where  $\xi$  and  $\eta$  are unit tangential vector fields on  $\partial\Omega$ .

For the estimates of mixed second order derivatives, we see from (5.2), (5.3) and (5.6) that

$$\mathcal{L}(\Psi \pm T_\alpha(u - \varphi)) \leq 0 \text{ in } \Omega_\delta$$

and

$$\Psi \pm T_\alpha(u - \varphi) \geq 0 \text{ on } \partial\Omega_\delta$$

for  $\alpha < n$ . It follows that

$$(5.33) \quad |u_{\xi\nu}|_{C^0(\partial\Omega)} \leq C,$$

where  $\xi$  is any unit tangential vector on  $\partial\Omega$  and  $\nu$  is the unit inner normal of  $\partial\Omega$ . It suffices to establish an upper bound for the double normal derivative on the boundary  $\partial\Omega$  since  $(1 + n\gamma)\Delta u \geq 0$ .

As in [11], let  $T = \{T_i^j\}$  be a skew-symmetric matrix, such that  $e^T$  is orthogonal, where  $T_i^j$  is the entry of  $i^{th}$  row and  $j^{th}$  column of  $T$ . Let  $\tau = (\tau_1, \dots, \tau_n)$  be a vector field in  $\Omega$  given by

$$\tau_i = T_i^j x_j + a_i, \quad i = 1, \dots, n,$$

where  $a_i$  is a constant. Denote  $u_{\tau\tau} = \tau_i \tau_j u_{ij}$  and  $u_{(\tau)(\tau)} = (u_\tau)_\tau = \tau_i \tau_j u_{ij} + (\tau_i)_j \tau_j u_i$ . Similar to Lemma 2.1 of [11] we can prove the following lemma.

**Lemma 5.2.** *We have*

$$\mathcal{L}(u_{(\tau)(\tau)}) \geq (F[U])_{(\tau)(\tau)}.$$

*Proof.* Similar to Lemma 2.1 of [11], by the skew-symmetry of  $T$ , we have

$$(5.34) \quad F^{ij}(T_i^k u_{kj\tau} + T_j^k u_{ki\tau}) = -F^{ij,st}(T_i^k u_{kj} + T_j^k u_{ki})U_{st\tau}$$

and

$$(5.35) \quad \begin{aligned} & F^{ij}(2T_i^k T_j^l u_{kl} + T_i^k T_k^l u_{lj} + T_j^k T_k^l u_{li}) \\ &= -F^{ij,st}(T_i^k u_{kj} + T_j^k u_{ki})(T_s^k u_{kt} + T_t^k u_{ks}). \end{aligned}$$

Note that

$$(u_{(\tau)(\tau)})_{ij} = u_{ij(\tau)(\tau)} - 2T_i^k u_{kj\tau} - 2T_j^k u_{ki\tau} + 2T_i^s T_j^t u_{st} + T_j^t T_t^s u_{si} + T_i^t T_t^s u_{sj}.$$

We find

$$(5.36) \quad \begin{aligned} \mathcal{L}(u_{(\tau)(\tau)}) &= F^{ij}u_{ij(\tau)(\tau)} + \gamma \sum F^{ii}(\Delta u)_{(\tau)(\tau)} \\ &+ F^{ij}\left(2T_i^s T_j^t u_{st} + T_j^t T_t^s u_{si} + T_i^t T_t^s u_{sj} - 2T_i^k u_{kj\tau} - 2T_j^k u_{ki\tau}\right) \\ &+ \gamma \sum F^{ii}\left(2T_l^s T_l^t u_{st} + 2T_l^t T_t^s u_{sl} - 4T_l^k u_{kl\tau}\right). \end{aligned}$$

Next, since  $T$  is skew-symmetric,

$$2T_l^s T_l^t u_{st} + 2T_l^t T_t^s u_{sl} - 4T_l^k u_{kl\tau} = 0.$$

It follows from (5.34), (5.35) and (5.36) that

$$\begin{aligned} \mathcal{L}(u_{(\tau)(\tau)}) &= F^{ij}u_{ij(\tau)(\tau)} + \gamma \sum F^{ii}(\Delta u)_{(\tau)(\tau)} \\ &- F^{ij,st}(T_i^k u_{kj} + T_j^k u_{ki})(T_s^k u_{kt} + T_t^k u_{ks}) \\ &+ F^{ij,st}\left((T_i^k u_{kj} + T_j^k u_{ki})U_{st\tau} + (T_s^k u_{kt} + T_t^k u_{ks})U_{ij\tau}\right). \end{aligned}$$

Note that

$$(u_\tau)_{ij} = u_{ij\tau} - T_i^k u_{kj} - T_j^k u_{ki}.$$

We obtain

$$\begin{aligned} \mathcal{L}(u_{(\tau)(\tau)}) &= (F[U])_{(\tau)(\tau)} - F^{ij,st}(\gamma \delta_{ij}(\Delta u)_\tau + (u_\tau)_{ij})(\gamma \delta_{st}(\Delta u)_\tau + (u_\tau)_{st}) \\ &\geq (F[U])_{(\tau)(\tau)}. \end{aligned}$$

□

Now we establish the double normal derivative estimates. We may assume the origin is a boundary point such that  $e_n = (0, \dots, 0, 1)$  is the unit inner normal there and denote

$$M = \sup_{x \in \partial\Omega} D_{\nu\nu}u(x).$$

where  $\nu$  is the unit inner normal of  $\partial\Omega$  at  $x \in \partial\Omega$ . Without loss of generality, we assume

$$M = \sup_{\partial\Omega} |D^2 u|,$$

and

$$(5.37) \quad \sup_{\Omega} |D^2 u| \leq CM,$$

for some uniform constant  $C \geq 1$ . By Lemma 5.2, we have

$$(5.38) \quad \mathcal{L}(T_\alpha^2 u) \geq T_\alpha^2(F[U]) = T_\alpha^2(\psi + \varepsilon\eta(\psi)) \geq -C.$$

According to [11], we see

$$(5.39) \quad w(x) \equiv T_\alpha^2 u(x) - T_\alpha^2 u(0) \leq C_0(|x'|^2 + M|x'|^4) \equiv h(x')$$

for  $x \in \partial\Omega$  with  $|x'| \leq r_0$ .

Let

$$\overline{\Psi} = \frac{1}{\delta^4} \left( A_1(u - \underline{u}) + td - \frac{N}{2}d^2 + A_3|x|^4 \right) - A_2(u - \underline{u}) - \sum_{l \leq n} |\nabla_l(u - \varphi)|^2.$$

Using the same arguments to Theorem 5.1, by (5.38), (5.37) and (5.39), we can show that there exists positive constants  $A_1, A_2, A_3, N$  sufficiently large and  $t, \delta$  sufficiently small such that

$$\mathcal{L}(w - h(x') - M\overline{\Psi}) \geq 0 \quad \text{in } \Omega_\delta$$

and

$$w - h(x') - M\overline{\Psi} \leq 0 \quad \text{on } \partial\Omega_\delta.$$

It follows from the maximum principle that

$$(5.40) \quad w - h(x') - M\overline{\Psi} \leq 0 \quad \text{on } \bar{\Omega}_\delta.$$

It follows that

$$(5.41) \quad w \leq CM(u - \underline{u} + d) + CM|x|^4 + C \quad \text{on } \bar{\Omega}_\delta.$$

Therefore, for each small  $\sigma > 0$ , we can find a positive constant  $\delta_1^4 = C\sigma < \delta^4$  such that

$$(5.42) \quad w \leq CM(u - \underline{u} + d) + \frac{\sigma}{2}M + C \quad \text{on } \bar{\Omega}_{\delta_1}$$

and

$$(5.43) \quad \mathcal{L}h \leq (\sqrt{\sigma}M + C) \sum F^{ii} \quad \text{on } \bar{\Omega}_{\delta_1}.$$

Next, there exists a positive constant  $\delta_2 < \delta_1$  such that

$$C(u - \underline{u} + d) \leq \frac{\sigma}{2} \quad \text{on } \Omega - \hat{\Omega}_{\delta_2},$$

where  $\hat{\Omega}_{\delta_2} \equiv \{x \in \Omega : \text{dist}(x, \partial\Omega) > \delta_2\}$ , since  $|D(\underline{u} - u)| \leq C$  independent of  $\varepsilon$ . Hence we can derive from (5.42) that

$$(5.44) \quad w \leq \sigma M + C \quad \text{on } \bar{\Omega}_{\delta_1} \cap (\Omega - \hat{\Omega}_{\delta_2}).$$

On the other hand, by (4.1), there exists a positive constant  $C$  depending on  $\gamma^{-1}$ ,  $\delta_2$  and  $|u|_{C^1(\bar{\Omega})}$  such that

$$(5.45) \quad |w| \leq C \quad \text{in } \hat{\Omega}_{\delta_2}.$$



Thus, there exists a positive constant  $C_\sigma$  depending on  $\sigma$  and other known data such that

$$|w| \leq \sigma M + C_\sigma \quad \text{on } \bar{\Omega}_{\delta_1}.$$

Similar to Theorem 5.1, we can find positive constants  $A_1, A_2, A_3, t$  and  $N$  such that

$$\mathcal{L}(w - (\sigma M + C_\sigma)\Psi') \geq 0 \quad \text{in } \Omega_{\delta_1}$$

and

$$w - (\sigma M + C_\sigma)\Psi' \leq 0 \quad \text{on } \partial\Omega_{\delta_1} \cap \Omega,$$

where

$$\Psi' = \frac{1}{\delta_1^2} \left( A_1(u - \underline{u}) + td - \frac{N}{2}d^2 + A_3|x|^2 \right) - A_2(u - \underline{u}) - \sum_{l < n} |T_l(u - \varphi)|^2.$$

Note that the main terms to control  $\sum F^{ii}$  in Theorem 5.1 are  $u - \underline{u}$  and  $\frac{N}{2}d^2$ . We may assume that  $\sigma$  is sufficiently small. Reviewing the proof of Theorem 5.1, we can find constants  $t'$  sufficiently small and  $N'$  sufficiently large such that

$$(5.46) \quad \mathcal{L}(u - \underline{u} + t'd - \frac{N'}{2}d^2) \leq -\varepsilon_1 \sum F^{ii} \quad \text{in } \Omega_{\delta_1}$$

for some positive constant  $\varepsilon_1$  and

$$(5.47) \quad u - \underline{u} + t'd - \frac{N'}{2}d^2 \geq 0 \quad \text{on } \bar{\Omega}_{\delta_1}.$$

By (5.43), (5.46) and (5.47), we can choose a constant  $A$  sufficiently large such that

$$\mathcal{L}(w - (\sigma M + C_\sigma)\Psi' - A(\sqrt{\sigma}M + C)w - h) \geq 0 \quad \text{in } \Omega_{\delta_1},$$

and

$$w - (\sigma M + C_\sigma)\Psi' - A(\sqrt{\sigma}M + C)w - h \leq 0 \quad \text{on } \partial\Omega_{\delta_1},$$

where  $w = u - \underline{u} + t'd - \frac{N'}{2}d^2$ . Thus, by the maximum principle again, we have

$$w \leq (C\sqrt{\sigma}M + C_\sigma)(u - \underline{u} + d + |x|^2) + h(x') \quad \text{on } \bar{\Omega}_{\delta_1}.$$

Therefore we obtain

$$(5.48) \quad (T_\alpha^2 u)_n(0) \leq C\sqrt{\sigma}M + C_\sigma \quad \text{for each } \alpha < n.$$

It follows that

$$u_{n(\xi)(\xi)} \leq C\sqrt{\sigma}M + C_\sigma \quad \text{on } \partial\Omega$$

for any tangential unit vector field  $\xi$  on  $\partial\Omega$ .

Now choose a new coordinate system and suppose the maximum  $M$  is attained at the origin  $0 \in \partial\Omega$ , and near the origin  $\partial\Omega$  is given by (5.1). By the Taylor expansion, we have

$$u_n(x) \leq u_n(0) + \sum_{\alpha < n} u_{n\alpha}(0)x_\alpha + (C\sqrt{\sigma}M + C_\sigma)|x'|^2$$

for  $x \in \partial\Omega$  near the origin, where  $u_{n\alpha}(0)$  is bounded by (5.33). Denote

$$g \equiv u_n(x) - u_n(0) - \sum_{\alpha < n} u_{n\alpha}(0)x_\alpha - (C\sqrt{\sigma}M + C_\sigma)|x'|^2.$$

In (5.5), we may choose another group of positive constants  $A_1, A_2, A_3, t, N$  and  $\delta$  such that

$$\mathcal{L}(g - (\sqrt{\sigma}M + C_\sigma)\Psi) \geq 0 \quad \text{in } \Omega_\delta$$

and

$$g - (\sqrt{\sigma}M + C_\sigma)\Psi \leq 0 \text{ on } \partial\Omega_\delta.$$

Applying the maximum principle again we obtain

$$M = u_{nn}(0) \leq C\sqrt{\sigma}M + C_\sigma.$$

Choosing  $\sqrt{\sigma} < 1/2C$ , we get a bound  $M \leq C$  and (2.1) is proved.

*Remark 5.3.* We remark that in this paper, the condition that  $\gamma > 0$  is only used to establish the interior estimate (4.1).

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